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NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

VOL. XVII BATON ROUGE, LA., MAY, 1943 No. 8

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VOL. XVII BATON ROUGE, LA., May, 1943 No. 8

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.
The publication of Volume XVII is aided by an appropriation from the Jacob Houck Memorial Fund of the M. A. of A.

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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

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2. The name of the Chairman of each committee is the first in the list of the committee.

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Once More---Mathematics and the Engineer

Several days after having received the longer of the following letters from Mr. Stanley U. Benscoter the Editor-and-Manager requested his permission to use this letter in whatever manner might appear promising for the cause of a more effective serving of the engineer by mathematics. In reply to this request came the briefer of the two communications. This briefer note is valuable not only because it conveys the permission requested, but also for the guidance it tenders the Magazine administration in any program of mathematical aid to the engineer which it might undertake. Mr. Benscoter's reply to our request is as follows:

Dear Sir:

You may use my letter in any way that you wish. You will, no doubt, receive other similar letters. You may be able to take excerpts from these letters to illustrate the attitude of engineers toward your projected movement and to illustrate the type of problems in which engineers are interested. These letters may mention problems which engineers have solved by using mathematics and problems which they have not been able to solve. From the solved problems you might be able to make requests for written papers describing the solutions. The unsolved problems might be described in a published article which should prove to be interesting to mathematicians in showing the type of mathematical research which would be useful to engineers. An applied mathematics section of your magazine should be useful to college professors who are teaching mathematics to engineering students. I have forwarded the reprints which you sent me to several engineers who may be interested.

Yours truly,
Stanley U. Benscoter.

The letter of Mr. Benscoter, the publication of which in this Magazine is approved by him in the above note, now follows. For the convenience of the mathematical reader who might desire to refer

to a particular one of the several engineering problems proposed by Mr. Benscoter we have assumed the editor's privilege of numbering the problems as (1), (2), etc.

THE ENGINEER BOARD
BRIDGE SECTION
FT. BELVOIR, VA.
May 11, 1943.

Professor S. T. Sanders
P. O. Box 1322
Baton Rouge, La.

DEAR SIR:

The discussions in the latest issue of your NATIONAL MATHEMATICS MAGAZINE concerning applied mathematics were very interesting to me. I would like to offer my encouragement to you in the publishing of articles on applied mathematics as related to engineering problems. I believe that I know several engineers who would be interested in subscribing to the magazine if such a department should be established. It might be well to discuss with the editor of the new *Quarterly of Applied Mathematics* the relative grade of mathematics which you would publish. This cooperation would prevent an overlapping of efforts. I think that the average practicing engineer will be unable to read, with understanding, the majority of articles which appear in the *Quarterly of Applied Mathematics* and that there is still an open field in the United States for the publishing of articles in applied mathematics of a collegiate grade. I believe that you would really be surprised if you could know how many engineers cannot set up and solve a quadratic equation or how many will go to great lengths to avoid the process. I am very much interested in the mathematics of engineering and will describe a few cases which I have met that may show you what is needed.

(1) Last year I was working on the design of several airports. The men who were computing surface elevations of the slabs ran into cases which they could not solve at the intersections of the runways. In each case I found that geometry and trigonometry led to a quadratic equation. Some interesting problems of this type could be published. The most hopeless unsolved problem which arose in laying out an airport consisted of trying to plan the movement of the earth from the high spots to the low spots in such a manner as not to exceed maximum permissible grades on the runways and to have a minimum amount of total earth movement. Many hours were wasted in trial and error procedures to accomplish this result and I believe that some

logical procedure could be devised for the problem. Since an airport slab covers a large area, its thickness should be determined accurately at least to the nearest inch. Yet the only method available is an approximate one given by Professor H. M. Westergaard in 1925. Engineers to not understand the mathematics behind his formulas but Professor Westergaard put his solution in such a form that engineers could read and use it.

(2) In 1938, at the Fifth Conference on Applied Mechanics, Professor D. L. Holl gave a solution of the elastic plate on an elastic foundation which might be very useful in developing better methods of stress analysis for airport slabs. However, the solution makes use of Fourier-Bessel transforms and the results are expressed in terms of infinite integrals. Engineers do not understand these things and hence this solution has not found the practical application which it deserves. The solution needs to be translated into engineering language.

(3) The most common occurrence of the cubic equation that I have found is in the stress analysis of a reinforced concrete member subjected to thrust and bending. I have taught number of my coworkers to face this problem and solve it by trial and error rather than shying away from it as they will invariably do without encouragement. Mathematical research is still needed to develop a method whereby a designer can choose a member to be in accord with standard specifications and at the same time be sure that he is using only a minimum amount of required reinforcing steel.

(4) In the December, 1940, issue of the *Journal of Applied Mechanics* Professor R. Oldenburger wrote an article on the convergence of the method of moment distribution for solving indeterminate structures. From ten years of use I know that in all practical cases the process of moment distribution does converge and so I was not much concerned with this proof. However, it was accomplished by using matrices which was something new to me. This paper gave me the conviction that it might be possible to use matrices in developing a simple expression for the complete solution of a complicated structure. In order to learn about matrices I studied for a half hour each day for a year in two small books. I went through the proofs of many theorems and worked out many examples. When I had finished this study I realized that the only things which I had learned of practical value were the processes of addition, subtraction, multiplication and division of matrices. I immediately forgot all of the theorems and their proofs but remembered how to perform the elementary operations. I have written a short statement of how these operations are performed which I believe any engineer could read and put into practice in a few

hours. It is discouraging to have to go through a great deal of theoretical mathematics in order to find a small amount which can be put to use. I have found several valuable applications of matrices in my work during the past year which might be interesting for publication. I have learned to express the solution of any indeterminate structure in terms of matrices and am now studying the method to see if it is practically usable.

(5) At the Fifth Conference on Applied Mechanics Professor S. Timoshenko presented a solution for the fixed-edge rectangular plate. The special case of a square plate with uniform load resolved into the process of solving an infinite number of equations in an infinite number of unknowns. These unknowns were the coefficients of a Fourier series. He showed that sufficient accuracy could be obtained by using only five equations to determine five unknowns. The exact solution of this problem would require the reciprocal of an infinite matrix. I have not been able to find any method of computing the inverse of an infinite matrix and I think that some published discussion of this problem might be useful.

(6) Recently an article of mine was published in the *Journal* of the American Concrete Institute giving formulas for bending moments in a certain type of structure. Quite a bit of algebra and calculus was used in deriving these formuals but none of the mathematics was published. I merely gave the formulas and stated the theory on which they were based. The editors did not even suggest that the mathematics might be shown, as they knew that a great majority of the readers would not care to follow through the mathematics. A minority of the readers might like to see the mathematics. This causes me to wonder if you could not publish the mathematics behind certain engineering articles which would otherwise be lost. Some of the engineering journals might be interested in discussing the possibilities of this idea. The Structural Bureau of the Portland Cement Association might have some suggestions along this line as they do quite a bit of mathematical work.

I hope that this letter will convey some useful suggestions.

Yours truly,

STANLEY U. BENSCOTER,
Associate Engineer.

UNITED STATES MARINE CORPS
MARINE CORPS OFFICER PROCUREMENT OFFICE
LOUISIANA BUILDING, 217-227 CAMP STREET
NEW ORLEANS, LOUISIANA

U. S. Marine Corps Women's Reserve

(For Immediate Release)

Women of America. The opportunity to serve your country in the uniform of the United States Marine Corps is now open to you. Selection of members for the U. S. Marine Corps Women's Reserve is underway. Never have American women faced a greater challenge than in this war, and never have they met a challenge with greater spirit. If you feel *you* are not giving your best, you can be sure of doing *your full part* when you join the U. S. Marine Corps Women's Reserve.

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For more complete information, write or call the U. S. Marine Corp Officer Procurement Office, 217 Camp Street, New Orleans, La., or your nearest Marine Corps Recruiting Station.

Euclidean Concomitants of the Triangle

By T. L. WADE
University of Alabama

1. *Introduction.* In this paper we use the results of Cramlet[1] and the writer[2] [3] on the construction and reduction of algebraic concomitants with the use of tensor algebra in the study of the euclidean concomitants of the triangle considered as a three-line configuration. Considerable use is made of numerical tensors, of which the ϵ-systems discussed by H. V. Craig [4] in his recent paper in this MAGAZINE are instances. Since we are working in the plane, or the (homogeneous) ternary domain, a free index is to have the values 1,2,3, and a repeated index is to be summed for these values. We take for the equations of the sides of the triangle under consideration

$$B_iX^i=0, \qquad C_iX^i=0, \qquad D_iX^i=0. \tag{1}$$

From Theorem III of [2] we have

Theorem I. *Every euclidean concomitant for the set of ground forms* (1) *is expressible by composition as a tensor of order zero with the use of the coefficient tensors* B_i, C_i, D_i, *the coordinate tensors* X^i *and* $U_i(U_iX^i=0)$, *and the numerical tensors* ϵ^{ijk}, L_i, *and* E^{ij}.

ϵ^{ijk} is the well known skew-symmetric numerical tensor, and

$$L_i=[0,\quad 0,\quad 1]\ , \qquad E^{ij}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is convenient to introduce the tensor A^{ij}, where*

$$A^{ij}=\epsilon^{ijk}L_k=\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

*A tensor is an array of scalars which obey a certain transformation law when the variable coördinates are transformed in a given manner. If the coördinate transformation is $x^i=t_k^{\ i}x^{-k}$, a 2-way matrix is a contravariant tensor of order two if its components satisfy $M^{ij}=|t|^{-w}t_k^{\ i}t_l^{\ j}\,\overline{M}^{kl}$, for some constant w, where $|t|$ is the determinant of the transformation matrix $t_k^{\ i}$. Once the tensorial character of the matrix M^{ij} is established, it may equally well be denoted by the use of any other indices, as M^{ab}.

A numerical tensor is one whose components are constants in all coördinate systems. In 2-dim. projective geometry, using homogeneous point coördinates $x^i=x^1,\ x^2,\ x^3$, the

As a basis for the reduction of the concomitants constructed in accordance with Theorem I, and for the investigation of the syzygies on them, we have from Theorem II of[3]

Theorem II. *Every identity satisfied by the concomitants constructed by Theorem I can be established with the aid of eleven basic identities.* All of these identities are given in explicit form in[3]; in this paper we use only two of them, the ones referred to in[3] as Iden. 4 and Iden. 5. They are

Iden. 1 $$A^{ij}A^{kl}\equiv E^{ik}E^{jl}-E^{il}E^{jk}.$$

Iden. 2 $$A^{ij}E^{kl}\equiv A^{kj}E^{il}+A^{ik}E^{jl}.$$

2. *An irreducibly complete system of euclidean concomitants for the triangle.* We should keep in mind the skew-symmetry of ϵ^{ijk} and of A^{ij}. As a consequence of this skew-symmetry many formal invariants vanish identically; among such is $B_iB_jB_k\epsilon^{ijk}$. Using Theorem I it is found that the line $B_iX^i=0$ has the seven non-vanishing concomitants

(2) $$B_iB_jE^{ij},\ B_iU_jE^{ij},\ B_iU_jA^{ij},\ B_iX^i,\ L_iX^i,\ U_iU_jE^{ij},\ U_iX^i.$$

These seven concomitants form an irreducibly complete system for the line. For certainly no one of them is rationally and integrally expressible in terms of the others, because there is no one of the identities of Theorem II applicable to any one of them. And from Theorem I we know that any other concomitant is a polynomial in terms of them.

The irreducibly complete system for three lines consists of their particular systems and their simultaneous system. Using the argument of the above paragraph, it is found that the simultaneous system of the lines $B=B_iX^i=0$ and $C=C_iX^i=0$ consists of

(3) $$B_iC_jA^{ij},\qquad B_iC_jE^{ij},\qquad B_iC_jU_k\epsilon^{ijk}.$$

The lines C and D have three simultaneous concomitants corresponding to (3), and so do the lines B and D. All three lines have only one simultaneous concomitant, namely $B_iC_jD_k\epsilon^{ijk}$. With these considerations in mind we are led to

only relative numerical tensors[1] are ϵ^{ijk} and ϵ_{ijk}. Both are skew-symmetric in the three indices, and their components have the following values: 0, when any two of the indices are equal; $+1$ when i,j,k is an even permutation of 123; -1 when i,j,k is an odd permutation of 123. Affine geometry is characterized by the adjunction of L_i, and euclidean geometry by the adjunction of E^{ij}.[2]

Theorem III. *The irreducibly complete system of euclidean concomitants for the lines* (1) *consists of the twenty-five concomitants listed in the following table:*

IRREDUCIBLY COMPLETE SYSTEM FOR THE TRIANGLE

No.	*Type*	*Tensor Expression*	*Geometric Interpretation*
1	Invariants	$I_1 = B_i B_j E^{ij}$	If $I_1 = 0$, B is a minimal line.
2		$I_2 = C_i C_j E^{ij}$	If $I_2 = 0$, C is a minimal line.
3		$I_3 = D_i D_j E^{ij}$	If $I_3 = 0$, D is a minimal line.
4		$I_4 = C_i D_j A^{ij}$	If $I_4 = 0$, C and D are parallel.
5		$I_5 = D_i B_j A^{ij}$	If $I_5 = 0$, D and B are parallel.
6		$I_6 = B_i C_j A^{ij}$	If $I_6 = 0$, B and C are parallel.
7		$I_7 = C_i D_j E^{ij}$	If $I_7 = 0$, C and D are perpendicular.
8		$I_8 = D_i B_j E^{ij}$	If $I_8 = 0$, D and B are perpendicular.
9		$I_9 = B_i C_j E^{ij}$	If $I_9 = 0$, B and C are perpendicular.
10		$I_{10} = B_i C_j D_k \epsilon^{ijk}$	If $I_{10} = 0$, B, C, D are copunctual.
11	Covariants	$\mathfrak{L} = L_i X^i$	$\mathfrak{L} = 0$ is eq. of line at infinity.
12		$B = B_i X^i$	The ground forms.
13		$C = C_i X^i$	
14		$D = D_i X^i$	
15	Contravariants	$\omega = U_i U_j E^{ij}$	$\omega = 0$ is eq. of absolute.
16		$P_1 = C_i D_j U_k \epsilon^{ijk}$	$P_1 = 0$ is eq. of pt. of inters. of C and D.
17		$P_2 = D_i B_j U_k \epsilon^{ijk}$	$P_2 = 0$ is eq. of pt. of inters. of D and B.
18		$P_3 = B_i C_j U_k \epsilon^{ijk}$	$P_3 = 0$ is eq. of pt. of inters. of B and C.
19		$Q_1 = B_i U_j E^{ij}$	$Q_1 = 0$ is eq. of pt. at infinity in the direction perpendicular to B.
20		$Q_2 = C_i U_j E^{ij}$	$Q_2 = 0$ is eq. of pt. at infinity in the direction perpendicular to C.
21		$Q_3 = D_i U_j E^{ij}$	$Q_3 = 0$ is eq. of pt. at infinity in the direction perpendicular to D.
22		$R_1 = B_i U_j A^{ij}$	$R_1 = 0$ is eq. of pt. at infinity on B.
23		$R_2 = C_i U_j A^{ij}$	$R_2 = 0$ is eq. of pt. at infinity on C.
24		$R_3 = D_i U_j A^{ij}$	$R_3 = 0$ is eq. of pt. at infinity on D.
25	Mixed Concom.	$U = U_i X^i$	The universal concomitent.

3. *Syzygies on the invariants of the triangle.* Irreducible concomitants are in general too numerous to be algebraically independent. In case there are $p+q$ irreducible concomitants, p of which are algebraically independent, there are q functionally independent syzygies which bind together the irreducible concomitants. The general equations of the three lines contain nine coefficients. The three euclidean trans-

formation parameters eliminated between the nine equations expressive of the identity of old and new forms leave six equations connecting old coefficients. So the ten irreducible invariants must be connected by four independent syzygies. We have using Iden. 1

$$I_4^2 \equiv C_iD_jA^{ij}\cdot C_kD_lA^{kl} \equiv C_iD_jC_kD_l(A^{ij}A^{kl}) \equiv C_iD_jC_kD_l(E^{ik}E^{jl}-E^{il}E^{jk}),$$

or S_1: $$I_4^2 \equiv I_2I_3 - I_7^2.$$

Similarly S_2: $$I_6^2 \equiv I_1I_2 - I_9^2.$$

By Iden. 2

$$I_9I_4 \equiv B_iC_jE^{ij}C_kD_lA^{kl} \equiv B_iC_jC_kD_l(A^{kl}E^{ij}) \equiv B_iC_jC_kD_l(A^{il}E^{kj}+A^{kl}E^{ij}),$$

or S_3: $$I_9I_4 = -I_5I_2 - I_6I_7, \quad \text{or} \quad I_9I_4 + I_5I_2 + I_6I_7 \equiv 0.$$

Similarly S_4: $$I_7I_5 + I_6I_3 + I_4I_8 \equiv 0.$$

By means of the syzygies S_1, S_2, S_3, S_4 the nine invariants $I_1, \cdots, I_9$ can be expressed algebraically in terms of the five I_1, I_2, I_4, I_7, and I_9. Moreover, these last five invariants are algebraically independent, for the matrix.

$$\left(\frac{\partial(I_1, I_2, I_4, I_7, I_9)}{\partial(B_1, B_2, C_1, C_2, D_1, D_2)} \right)$$

has rank five. The invariant I_{10} is clearly independent of these five; for B_3, C_3, D_3 appear only in I_{10}. So we have

Theorem IV. *Of the ten irreducible invariants $I_1, \cdots, I_{10}$ for the triangle six are algebraically independent, and these may be chosen as I_1, I_2, I_4, I_7, I_9, and I_{10}. The ten irreducible invariants are connected by four functionally independent syzygies, and these may be chosen as S_1, S_2, S_3, S_4.*

4. *Geometric absolute invariants and polynomial concomitants in terms of members of the irreducibly complete system.* Let Θ_i denote the angle at $P_i (i = 1,2,3)$. It is readily verified that (It is assumed here and in the remainder of the paper that none of the invariants are zero.)

$$\sin^2\Theta_1 = \frac{I_4^2}{I_2I_3}, \qquad \sin^2\Theta_2 = \frac{I_5^2}{I_3I_1}, \qquad \sin^2\Theta_3 = \frac{I_6^2}{I_1I_2},$$

$$\cos^2\Theta_1 = \frac{I_7^2}{I_2I_3}. \qquad \cos^2\Theta_2 = \frac{I_8^2}{I_3I_1}, \qquad \cos^2\Theta_3 = \frac{I_9^2}{I_1I_2}.$$

The well known formula for the square of the distance h of the point $Y^i=(Y^1, Y^2, Y^3)$ to the line $B_iX^i=0$,

$$h^2=\frac{(B_1Y^1+B_2Y^2+B_3Y^3)^2}{(Y^3)^2(B_1^2+B_2^2)} \tag{4}$$

may be written $$h^2=\frac{(B_iY^i)^2}{(L_iY^i)^2(B_iB_jE^{ij})}.$$

Let h_i denote the length of altitude drawn from P_i to the opposite side; d_i the length of the side of the triangle opposite P_i; R the radius of the circumcircle and Δ the area of the triangle. We have immediately from the second form of (4)

$$h_1^2=\frac{I_{10}^2}{I_4^2I_1}, \qquad h_2^2=\frac{I_{10}^2}{I_5^2I_2}, \qquad h_3^2=\frac{I_{10}^2}{I_6^2I_3}.$$

Using $d_1^2=h_2^2\csc^2\Theta_3$, and the two similar relations, we get

$$d_1^2=\frac{I_{10}^2I_1}{I_5^2I_6^2}, \qquad d_2^2=\frac{I_{10}^2I_2}{I_6^2I_4^2}, \qquad d_3^2=\frac{I_{10}^2I_3}{I_4^2I_5^2}.$$

Thence $$\Delta^2=\frac{1}{4}d_i^2h_i^2=\frac{1}{4}\ \frac{I_{10}^4}{I_4^2I_5^2I_6^2};$$

$$R^2=\frac{1}{16}\ \frac{(d_1\cdot d_2\cdot d_3)^2}{\Delta^2}=\frac{1}{4}\ \frac{I_{10}^2I_1I_2I_3}{(I_4I_5I_6)^2}.$$

By Theorem I every polynomial concomitant of the triangle is expressible as a rational integral function of members of the irreducibly complete system. We now express some of the better known concomitants in this form.

The circumcircle. In normalized trilinear coordinates the equation of the circle circumscribed to the triangle formed by the lines $p_i=0(i=1,2,3)$ is (where Θ_i is the angle opposite p_i[5])

$$g\equiv p_2p_3\sin\Theta_1+p_3p_1\sin\Theta_2+p_1p_2\sin\Theta_3=0.$$

Transforming to Cartesian coordinates by putting $p_1=\dfrac{B}{(I_1)^{1/2}}$, etc., we get as the desired equation

$$G\equiv I_1I_4CD+I_2I_5DB+I_3I_6BC=0.$$

The self-polar circle. In trilinear coordinates the equation of the self-polar circle is

$$h \equiv p_1^2 \sin 2\Theta_1 + p_2^2 \sin 2\Theta_2 + p_3^2 \sin 2\Theta_3 = 0.$$

With the transformation mentioned above this becomes

$$H \equiv I_4 I_7 B^2 + I_5 I_8 C^2 + I_6 I_9 D^2 = 0.$$

The nine-points circle. In trilinear coordinates the equation of the nine-points circle is $g - h = 0$. So in Cartesian coordinates it is $G - H = 0$.

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Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

A Sixth Lesson in the History of Mathematics

By G. A. MILLER
University of Illinois

In the preceding lesson, published in this journal, pages 212-220 of the present volume, we emphasized the very slow development of the theory of complex numbers and the long successful use of these numbers before a satisfactory theory of their use with respect to the fundamental operations of arithmetic had been published. This was done about the beginning of the nineteenth century. The need of a consideration of this matter on the part of the serious students of the history of mathematics may become clearer in view of the following quotation from E. T. Bell's *Development of Mathematics*, page 159 (1940): "The first clear recognition of imaginaries was Mahavira's extremely intelligent remark in the ninth century that in the nature of things a negative number has no square root. He had mathematical insight enough to leave the matter there and not to proceed to meaningless manipulations of unintelligible symbols."

On the contrary, the history of mathematics furnishes many instances, besides the one noted above, where important advances in mathematics were due to what were then meaningless manipulations of unintelligible symbols. The wise course in mathematics is to *experiment freely with meaningless symbols as long as the results thus obtained have always been found to be correct.* If mathematicians had continued to do what Mahavira is said to have done there would not have resulted, as far as we can see, the very important theory of complex numbers. Correct results have often led to a scrutiny of the methods employed with a view towards understanding these methods. Mahavira's course is an example of what should not be done in mathematics and deserves censure rather than praise.

14. *The circle and the sphere.* The histories of the circle and of the sphere like those of the solutions of the quadratic and the cubic equations extend through thousands of years beginning with ancient Babylonian and ancient Egyptian mathematics. Notwithstanding the great simplicity of these geometric figures their study has led to very serious difficulties which were not overcome before the latter part of the nineteenth century. This is partly due to the fact that the ratio between the circumference of a circle and its diameter is a transcendental number and these numbers were not proved to exist before about the middle of the nineteenth century, when the French mathematician, J. Liouville (1809-1882) proved definitely that such numbers actually exist and thus started a long series of very interesting investigations along this line.

While J. Liouville was the first to establish the existence of transcendental numbers he did not prove that the ratio between the circumference of a circle and its diameter is such a number. This fact was first proved by a German mathematician, F. Lindemann (1852-1939) during the year in which J. Liouville died. There is an interesting difference between the histories of the number which represents the ratio between the diagonal of a square and its side and the number which represents the ratio betweeen the circumference of a circle and its diameter. When the ancient Greeks first observed (about a hundred years after the death of Pythagoras) that the former ratio could not be a rational number they assumed that it was no number at all and were led to assume that there is a fundamental difference between numerical quantities and geometric quantities. This assumed difference greatly affected ancient Greek and medieval mathematics.

On the contrary it seems to have been always assumed that the ratio between the circumference of a circle and its diameter is a number, and a very large number of assumptions were made by the ancient and the medieval peoples as regards the size of this number. Among these supposed sizes there are those which are represented by the numbers 3, $3\frac{1}{7}$, $3\frac{1}{8}$, $\sqrt{10}$, and $3\frac{55}{113}$. The fact that the $\sqrt{10}$ is an irrational number does not imply that it was then assumed that the given ratio is an irrational number. The fact that it cannot be a rational number seems to have been first proved about 1766 by J. H. Lambert (1718-1777). The great simplicity of the circle and the sphere from a geometric point of view and the great difficulties which transcendental numbers involve illustrate how effectively geometry and algebra have supported each other in the development of our subject.*

*Hence the expression "Dilemma of Pythagoras" used on page 61 of the *Survey of Modern Algebra* by Birkhoff and MacLane (1941) is not in accord with the established facts of the history of mathematics at the present time as regards irrationals.

The term circle is used with two different meanings in Euclid's *Elements* and this ambiguous meaning of the term is still common in the mathematical literature. By circle we may mean a finite plane surface such that every point on its boundary is at the same distance from a fixed point within it which is called the center of the circle, or we may mean the curve which circumscribes this surface. Similarly, by a sphere we may mean a volume which is bounded by a surface all of whose points are at the same distance from a fixed point within the volume called the center of the sphere. These two meanings of the same term have persisted. While they occasionally are a source of vagueness the actual meaning can usually be inferred from the context. At any rate their continued existence through such a long period of time is a matter of historical interest.

A fact of considerable historical interest in regard to the circle is that the ancient Greeks did not use a special term corresponding to our term radius. They regarded the diameter of the circle as the fundamental straight line with respect to it and referred to this line the the other linear measures relating to the circle. On the contrary the later Indian mathematicians used a special word corresponding to our term radius. They may possibly have been influenced in this respect by the much earlier Babylonian mathematicians who seemed to have regarded as the fundamental straight line in the circle not the diameter, but the radius since they assigned to the latter their fundamental number 60. These facts are closely related to the early developments in trigonometry as regards the use of our ordinary trigonometric functions which are related to the radius. The term radius used to represent one half of the diameter of a circle seems to have originated in the sixteenth century but it was rapidly adopted thereafter.

The apparent shape of the sun and the full moon naturally directed attention to the circle in very early times. It is not known when the idea to compare the area of the circle with that of a square began to interest the human race but this took place long before the time of Euclid since the ancient Egyptians in the work of Ahmes, written about 1700 B. C., assumed that the area of the circle is equal to that of a square on $^{8}/_{9}$th of its diameter. We find here an assumption which implies that the areas of circles are to each other as the squares on their diameters and this assumption naturally suggested that similar figures are to each other as the squares on their homologuous sides. It took a long time before more accurate results were obtained in regard to the exact area of a circle. In Euclid's *Elements* we do not find the measurement of surfaces but only the comparison of their areas which is a somewhat more general point of view, but shortly

after Euclid's time Archimedes proved that the area of a circle is equal to that of a right triangle whose base is equal to the circumference of the circle and whose altitude is the radius of the circle.

The circle may be regarded as the most important closed figure of plane geometry and its study has given rise to various subjects. Its symmetry and its simple definition make a strong appeal to all who are interested in geometry. The fact that its area is larger than that of any other plane figure having the same perimeter surprises no one even if it is somewhat difficult to prove. The study of elementary trigonometry is largely based on that of the circular functions sine, cosine, tangent, cotangent, etc., which in turn, have led to the construction of a large number of tables giving approximate values of of these functions for different angles. They also led to a vast number of formulas exhibiting numerous interrelations between these functions, which are extremely used both in pure and in applied mathematics. Various subjects which lead to great difficulties when we restrict ourselves to accurate results become comparatively easy if we are satisfied with approximate values which are at least as accurate as the measurements to which they relate. The use of such numbers has a long history.

Just as the circle may be regarded as the most symmetrical among the figures of plane geometry so the sphere may be regarded as the most symmetrical among the figures of solid geometry and it also naturally attracted much attention among the people who first cultivated the study of mathematics, and it has since then been fundamental in the development of our subject. Euclid noted the fact that every plane section of the sphere is a circle. The fact that the area of a sphere is equal to that of four such largest plane sections seems to have been first definitely proved by Archimedes. It was announced by W. W. Struve in his translation of the *Moscow Mathematical Papyrus* (1930) that the result was stated therein. The correctness of this translation was, however, questioned later by T. E. Peet in the *Journal of Egyptian Archæology*, volume 57, page 100 (1938). Hence we do not know now that the ancient Egyptians even guessed this area correctly. A proof of a rule for finding this area was far beyond their mathematical attainments judging from their extant mathematical work.

The development of the geometry of the sphere was largely inspired by the facts that the stars appear to be situated on an approximately spherical surface and that some of the ancient Greeks already assumed that the earth has a spherical surface. In Euclid's *Elements* it is proved that spheres are to one another as the cubes of their re-

spective diameters and Menelaus (about 100 A. D.) treated spherical triangles in about the same way as Euclid had treated plane triangles in his *Elements*, but as regards congruent triangles new questions arose in view of the fact that when equal parts appear in the reverse order the spherical triangles cannot be superimposed as is possible with the plane triangles. A. M. Legendre (1752-1833) was the first to publish a proof of the fact that symmetrical spherical triangles have the same area and he introduced the concept of symmetrical congruent triangles. Menelaus had already noted that the sum of the angles of a spherical triangle exceeds two right angles.

Archimedes proved the beautiful theorem that the ratios between the volume and the area of a sphere and the corresponding functions of the circumscribed right cylinder is $^2/_3$, and it is reported that the tombstone on the grave of Archimedes contained the figure of a sphere inscribed in a right cylinder. Cicero is reported to have determined thereby (about 75 B. C.) the place where Archimedes was buried and to have provided for the renewal of the original surroundings. There existed also coins from the city of Syracuse which contained the same figure in honor of their noted former citizen. C. Ptolemy (about 150 A. D.) is the earliest important Greek writer on calculating spherical trigonometry whose works are now known. His work, commonly known by its Arabic name, as the *Almagest*, was for a long time a standard work on astronomy and corresponded somewhat to Euclid's *Elements* as a standard work on elements of mathematics. The spherical trigonometry in the *Almagest* aimed to be useful in the study of astronomy and not to be a special work on the abstract sphere. The Persian mathematician Nasir Eddin (1201-1274) seems to have been the first to develop spherical trigonometry as a subject by itself. He used the law of sines in what he called his modern method and gave several proofs of it. The older method was largely based on the transversal theorem of Menelaus.

Euclid noted that a sphere is generated by the rotation of a semicircle around its bounding straight line, and the study of Archimedes relating to the sphere inaugurated a number of questions in regard to rotation surfaces and solids. Heron (about 100 B. C.) calculated the volume of the solid generated by a circle when it is rotated around a straight line in its plane. About 500 years after Archimedes, Pappus proved a very general theorem which has often been regarded as equivalent to the modern theorem known as Guldin's theorem and appears in some of our textbooks on elementary calculus. In fact, Guldin has been called a plagiarist for publishing this theorem without giving due credit to Pappus but this view is not accepted by many

of the later writers on the subject. A footnote relating to this subject appears in C. B. Boyer's *Concepts of the Calculus*, page 139 (1939), mentions the differences of views on this subject, with references.

Although the circle and the sphere have such simple definitions it is amazing how many mathematical theorems relate to these figures. Two of the thirteen books of Euclid's *Elements* (III and IV) pertain largely to the geometry of the circle, and in modern times a large number of additional theorems relating to this figure have been added to those known at the time of Euclid. When analytic geometry began to be extensively developed by R. Descartes (1596-1650) and others, the equations of the circle and the sphere received prominent attention in view of their relative simplicity and great usefulness in other parts of the subject. In the later period of the rapid development of the theory of groups during the nineteenth century they again assumed a central position, so that their history extends through both ancient and modern mathematics. They furnish many simple illustrations of the concept of group and of other concepts.

One of the well known discoveries of the seventeenth century in regard to the circle is the alternating series due to G. W. Leibniz (1674), which is as follows:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots.$$

This is a spherical case of a theorem announced a few years earlier by James Gregory but is supposed to have been discovered independently by G. W. Leibniz, who derived it in a manner widely different from that employed by James Gregory. To emphasize these facts we note here that on page 441 of the valuable volume entitled *What is Mathematics?* by Richard Courant and Herbert Robbins (1941) this series is incorrectly credited to Euler and is here called "one of the most beautiful mathematical discoveries of the eighteenth century" (page 441). It should be noted that while the given series is very interesting it is not of great direct practical use since it converges too slowly.

15. *Regular polygons and regular polyhedrons.* The histories of the regular polygons and regular polyhedrons are very closely related to those of the circle and the sphere and extend through about an equally long period of time. The fact that the former figures can be inscribed in a circle and a sphere in such a way that the vertices of the polygons are situated on the circumference of a circle and the vertices of the polyhedrons are situated on the surface of a sphere exhibit the close relations between these figures and make it appear probable that they

have many properties in common. Included among these properties are their symmetries which suggest the close contact between mathematics and art. This must be noted as an effective influence in the development of our subject through its entire history. This relationship is very difficult to evaluate but it deserves at least some notice in every serious study of the development of mathematics. Mathematical history suffers much by failing to exhibit contacts with related subjects which furnished inspiration.

The simplest regular polygon is the equilateral triangle and this is the first figure of Euclid's *Elements*. The six movements (including the identity) which transform it into itself are not all commutative when they are combined in pairs. It is therefore interesting that the commutative law did not receive a special name until 1814, as was noted in §9, volume 14, page 241 of this MAGAZINE. It was not until the nineteenth century that stress was laid on the question whether operations are commutative or non-commutative when they are combined in pairs. Hence the first figure in Euclid's *Elements* suggested properties which failed to receive much attention for more than two thousand years. The fact that each of the angles of the equilateral triangle is equal to 60° may have some connection with the great emphasis which the ancient Babylonians placed on the number 60 in their numerical work.

The square and the cube have been extensively used as fundamental units of measure since the most ancient times of which we have any record. It should be noted that by means of these units numerical relations between geometric figures of different dimensions can be directly established. Arithmetic and geometry are thus seen to be closely connected and numerical relations between geometric magnitudes of different dimensions become apparent. Arithmetic and geometry are thus seen to be connected and the interplay between the developments of these two subjects has always been a great source of their growth. The history of mathematics can scarcely emphasize too much the incentives to advances which arithmetic and geometry provided for each other at all times.

The fact that the sides of a regular hexagon are equal to the radius of the circumscribed circle was doubtless observed very early and may have been partly responsible for the emphasis on the sexagesimal system of numeration and notation among the ancient Sumerians and the ancient Babylonians. At any rate, the sexagesimal system of numeration and notation is a prominent fact in the history of mathematics and the regular hexagon has points of contact therewith and is a fundamental geometric figure whose history cannot now be

traced to its origin. By joining the vertices of a regular hexagon with the center of the circumscribed circle there result six equilateral triangles and the number 6 plays a prominent role in the history of mathematics both in geometry and in arithmetic. It should be emphasized that parts of mathematics often grew out of special simple cases whose solutions inspired interest but were not foreseen to become a part of a great body of knowledge. For instance, the complex numbers.

While it was obvious that a regular polygon can always be inscribed in a circle the methods of doing this by elementary geometry soon led to difficulties which the ancient Greeks were not able to overcome. It was C. F. Gauss (1777-1855) who first discovered that a regular polygon of a prime number of sides can be inscribed in a circle by elementary geometry if and only if this prime number reduced by unity is a power of 2. This proof was based on algebraic considerations and hence it again exhibited the fundamental interplay between algebra and geometry and showed that since mathematics has two legs to stand on it would be unwise to try to limit it to standing on one. According to the noted proof by C. F. Gauss the regular polygon of the smallest prime number of sides which cannot be inscribed in a circle by elementary geometry is the regular heptagon and hence this figure naturally received considerable attention on the part of Archimedes and other Greek geometers.

The great interest which the ancient Greeks took in regular figures is reflected in the fact that the last book of Euclid's *Elements* is devoted to regular polygons and to the five possible regular convex polyhedrons, viz., the regular tetrahedron, the cube, the regular octahedron, the regular dodecahedron, and the regular icosahedron. These five solids have been called the *cosmic solids* or the *Platonic solids* and they are frequently noted in Greek philosophy as well as in their mathematics. In particular, fire, air, water and earth were supposed to be composed respectively of regular tetrahedrons, regular octahedrons, regular icosahedrons, and cubes; while the universe was supposed to be incclosed by a regular dodecahedron. Although these assumptions were without foundations they exhibit an interest at that time in these regular solids and tended to direct attention to their symmetries.

About the middle of the nineteenth century various mathematicians began to lay greater stress on the formal laws of operations. The discovery of quaternions and the study of permutation groups tended to emphasize the importance of the associative law in the combination of symbols even when the commutative law is not satis-

fied. In this movement an Irishman by the name of W. R. Hamilton (1805-1865) took a prominent part. On November 10, 1856, he presented a paper before the Royal Irish Academy in which he considered the equations

$$1 = i^2 = k^3 = l^r, \qquad ik = l, \qquad r = 3,4,5.$$

He observed that these equations are sufficient to restrict the elements i,k,l when they are combined with respect to multiplication, under the associative but not the commutative law, in every possible manner, so that the total number of the distinct possible products may be interpreted as movements of the regular polyhedrons. These facts can easily be verified.*

He thus exhibited what was then a new insight into the properties of the regular solids. For instance, when $r = 3$ there result the laws of combination of the twelve movements which transform the regular tetrahedron into itself. Since il is then an operation of order 3 it results that i and k are non-commutative but i and l^2il are commutative since $l^2ili = il^2il$. Hence i, l^2il, lil^2 are the three elements of period two in the so-called four group, which is invariant under the 12 distinct operations generated by i and k. Similarly, it may be proved that when $r = 4$ the twenty-four distinct elements generated by i and k obey the same laws of combination as the 24 movements which transform the cube into itself and when $r = 5$ these 60 elements are in a 1,1 correspondence with the 60 movements which transform a regular icosahedron into itself. The formal laws of these elements combined under the associative law thus exhibited about the middle of the nineteenth century, properties in regard to the cosmic solids which had not been noted before.

Euclid constructed in his *Elements* regular polygons of three, four, five, and fifteen sides. It is easy to construct a regular polygon having twice as many sides as those of a given regular polygon which has been constructed. Hence Euclid's methods showed how to construct regular polygons which have a number of sides equal to that given by each of the following expressions:

$$3 \cdot 2^n, \quad 4 \cdot 2^n, \quad 5 \cdot 2^n, \quad 15 \cdot 2^n.$$

The smallest numbers greater than 2 which are not given by one of these expressions are 7, 9, 11, 13, and 24. Efforts to find methods of construction by means of ruler and compasses of regular polygons having a number of sides equal to one of these exceptional numbers failed, and the two thousand years following Euclid produced neither

*Miller, Blichfeldt, Dickson *Finite Groups*, reprint 1938, page 143.

an increase in the number of the types of regular polygons which can be constructed in the given manner nor a theory exhibiting the real nature of the difficulty, as was noted above.

While C. F. Gauss furnished a definite answer to the question whether a regular polygon having a given prime number of sides can be inscribed in a circle by means of a ruler and compasses this answer, as is usual in mathematics, raised other and more difficult questions. Among these is the determination of the values of m for which a number of the form 2^m+1 is a prime number. Judging from the past the development of mathematics points to an endless growth of this subject since the solution of problems of long standing calls attention to other more difficult problems. It was observed very early that a floor can be completely covered by equilateral triangles, squares, or regular hexagons, but there is a long step from such an observation to an extensive study of the regular polygons. It should however, be emphasized that ornaments had a considerable part in the development of ancient mathematics and these ornaments frequently involved regular figures. Many of them appear in the *Grammar of Ornaments* by O. Jones, 1856, which has been widely used, especially in architecture. A. Sperier refers to this subject in the introduction to the second edition of his *Theorie der Gruppen* (1927) and makes some suggestions as regards the influence of ornaments in the development of ancient mathematics before the classical period of the Greeks.

A very interesting formula relating to polyhedrons is commonly known as Euler's polyhedron theorem but it was known earlier and it has been suggested that it may have been known already by Archimedes in view of the fact that Archimedes determined completely the so-called semi-regular polyhedrons. The formula is

$$c+f=e+2$$

when $c=$the number of corners, $f=$the number of faces, and $e=$the number of edges. The formula was announced by R. Descartes near the beginning of the seventeenth century and a number of different writers published supposed proofs of it, but in 1812 it was noted that the formula does not apply to every polyhedron and it was proposed that the polyhedrons which satisfy this formula might be known as Euler polyhedrons. Few other subjects have had such a long and powerful influence on the development of mathematics as that of the regular polygons and the regular polyhedrons, and in his *Entwisklung der Mathematik*, volume 1, page 358 (1926) Felix Klein included a brief eulogy on the regular bodies in view of their wide influence and motivating value in the entire history of our subject. Such concepts deserve special emphasis in this history, since they exhibit continuity of development during long periods of time.

The Teachers' Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT, J. S. GEORGES
and L. J. ADAMS

Pre-study Examinations in Mathematics*

By H. M COX
The University of Nebraska

I

The University of Nebraska has used classification examinations in freshman mathematics courses for the past ten years. Until 1940, the examination was administered alternatively in the Nebraska high schools or on the University campus. Beginning in September, 1940, the examination was included in a battery of pre-registration guidance examinations given to all entering freshmen.

Certain changes were necessary in the form of the examination in order to expedite scoring and in the scope of the examination in order to fit it for more extended use† in the educational and vocational guidance of students. The necessity for the rapid scoring of the University of Nebraska's guidance examinations is illustrated by the fact that from the four or five examinations administered to some 1500 students approximately 18,000 separate scores must be obtained, recorded, interpreted, and reported to registration advisers within a four-day period.

A short-matching type of question was selected for the mathematics examination to meet the requirements of rapid scoring and to partially offset the recognition factor.‡ Choices were given which necessitated computation and comprehension on the part of the student.

*A review of the 1940 Form of the Nebraska Mathematics examination which was prepared by Professors W. C. Brenke, H. M. Cox, and F. S. Harper.

†Cox, H. M., "What are the administrative and guidance uses of mathematics examinations?" *The American Mathematical Monthly*, Vol. 48, May, 1941, pp. 317-319.

‡In an experiment, a coefficient of correlation of .73 was obtained on 290 cases between the 1940 examination and an early form of the examination which used the completion type of question and which covered similar subject-matter.

TABLE 1—Distribution of scores on the 1940 classification examination in mathematics made by (1) all entering freshmen, (2) entering freshmen registered in three-hour college course in algebra, and (3) entering freshmen registered in five-hour college course in algebra and trigonometry. The University of Nebraska.

Scores	PART I, ARITH.			PART II, E. ALG.			PART III, A. ALG.			Scores	PART I+II			PART II+III		
	All	Math. 11	Math. 21	All	Math. 11	Math. 21	All	Math. 11	Math. 21		All	Math. 11	Math. 21	All	Math. 11	Math. 21
30	155	25	47	48	3	45	..	..	..	60	2	..	2	..	..	..
28-29	323	51	58	68	9	24	..	..	..	56-59	144	18	40	..	..	..
26-27	277	35	35	113	16	23	..	..	..	52-55	174	35	57	..	..	..
24-25	228	26	13	126	21	27	..	..	..	48-51	206	40	42	29	..	18
22-23	156	17	6	115	19	22	..	..	..	44-47	188	37	19	44	..	20
20-21	134	8	1	151	21	12	14	..	7	40-43	169	19	..	64	..	31
18-19	92	4	..	114	23	4	30	..	20	36-39	167	12	..	78	..	42
16-17	68	2	..	158	22	2	43	..	26	32-35	158	5	..	95	..	29
14-15	40	..	..	102	10	1	52	..	23	28-31	157	3	..	104	..	20
12-13	20	..	..	133	10	..	55	..	23	24-27	104	..	..	125	..	..
10-11	6	..	..	111	6	..	104	..	20	20-23	44	..	..	153	..	..
8-9	7	..	..	70	4	..	68	..	19	16 19	21	..	..	206	..	..
6-7	..	..	..	80	2	..	95	..	17	12-15	14	..	..	211	..	..
4-5	..	..	..	41	1	..	65	..	1	8-11	1	..	..	178	..	..
2-3	..	..	..	33	1	..	59	..	2	4-7	..	..	..	120	..	..
0-1	3	..	..	46	..	..	931	..	2	0-3	..	..	..	102	..	..
N. cases	1509	168	160	1509	168	160	1509	..	160		1509	168	160	1509	..	160
Median	26	26	33	17	19	24	1*	..	13		44	45	53	18*	..	33

*It has been assumed, in these calculations, that all students took the entire examination. However, students with less than three semesters of H. S. algebra took only Parts I+II. Median scores for students taking Part III are: Part III = 10; Part II+III = 33 (1940 data).

Questions were scaled from very easy to difficult: separate sections were provided for arithmetic computation and arithmetic problems, elementary high school mathematics (algebra and geometry), and advanced high school algebra. The three part scores and two combinations of part scores were obtained on the 1940 mathematics examination because proficiency levels were set independently by each undergraduate college. In three of the colleges, the proficiency levels were set so as to coincide with the critical levels established for admission to the freshman mathematics courses, but this was not the case in the other colleges.

II

About one-fourth of the students at the University of Nebraska take formal courses in mathematics. Two basic courses of study are provided. The first is a sequence of three-hour courses: elementary algebra, trigonometry, and analytic geometry (Math. 11, 12, and 13). The second is a sequence of two five-hour courses (Math. 21 and 22) covering these same fields, but restricted to students with more extended high school training and to students who demonstrate mathematical proficiencies of a high order as measured by the classification examination.

The mathematical attainments of the students registered in the two sequences of freshman courses can be illustrated by comparative distributions of scores made on the 1940 examination by these two groups relative to the distribution of scores made by all entering freshmen (see Table 1). It is to be noted that both groups of students registered for freshmen mathematics courses are superior to the average group of students. This average superiority relative to all University students is demonstrated on all of the pre-registration guidance examinations. Figure 1 shows the median scores made on the guidance examinations by the two groups of students registered for freshman mathematics courses.

Table 1 displays the distributions of raw scores for the parts of the mathematics classification examination and for the combinations of the parts. For purposes of classification, i. e., in determining the critical levels for admission to the freshman mathematics courses, these distributions are sufficient. But for the larger aspects of guidance, comparisons must be made between scores on mathematics and, say, English or science. To facilitate such comparisons, derived scores designated as *Nebraska scaled scores* are substituted for the raw scores. A score of *1* indicates that a student's score occurs in the highest 3% of the freshman class. A score of *5* indicates the middle 20% of the

distribution. The nine-point scale provides for equal linear measurements slightly in excess of one-half of the standard deviation of the *normalized* distribution.*

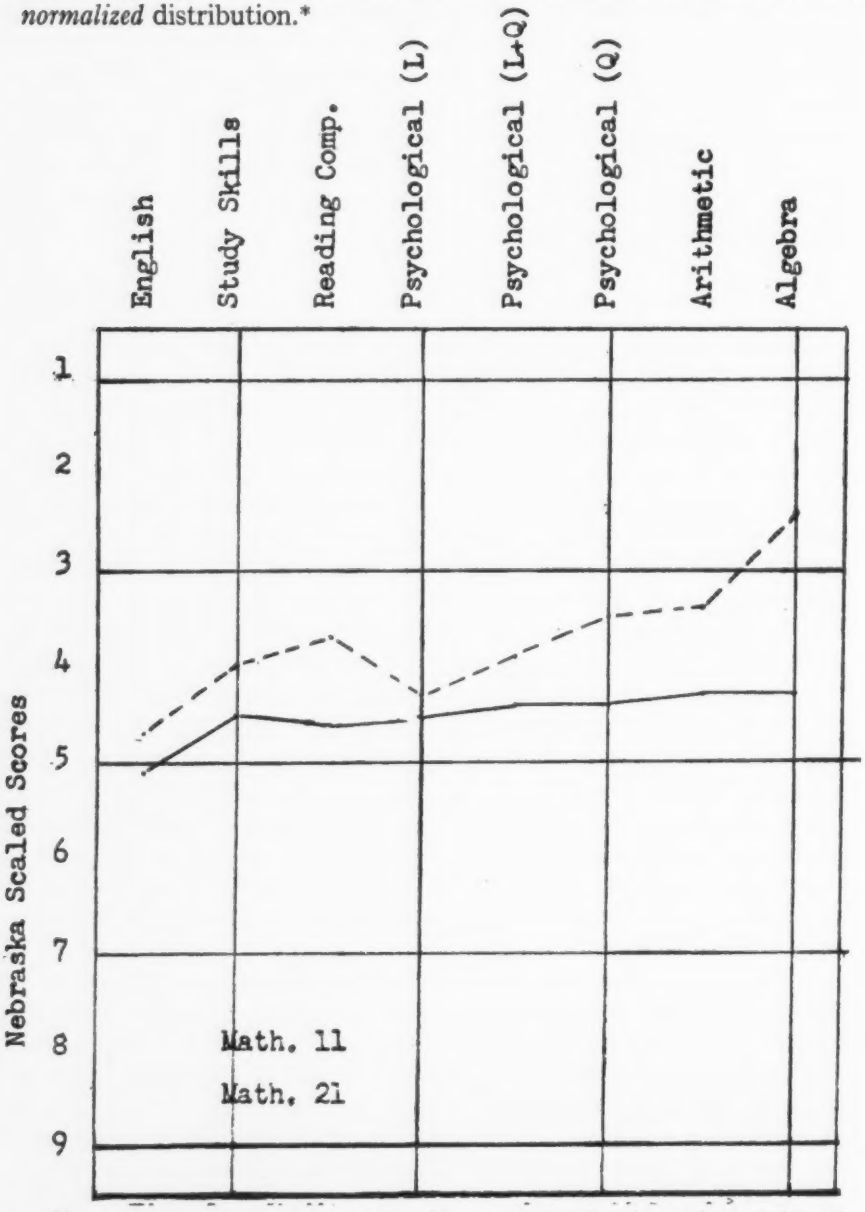

FIG. 1—Median scores made on the guidance examinations by students registered in Mathematics 11 and Mathematics 21.

*The percentages which result from the use of equal linear units are 3%, 7%, 12%, 18%, 20%, 18%, 12%, 7%, 3% for the *scaled scores* 1 to 9, respectively.

The derived score equivalents for raw scores are readily found by the use of a profile-chart (Fig. 2). The position of the raw score is located within a pair of limiting values and the *scaled score* is read from the chart in the column to the left. Thus a raw score of 30 on the Mathematics I+II examination is located between the scores 29 and 34; the corresponding scaled score is *7*, a score which falls in the lowest 22% of the distribution of scores made by entering freshmen.

THE UNIVERSITY OF NEBRASKA
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FORM F

NAME.....

Table of Scaled Score Equivalents for Pre-Registration Guidance Examinations, September, 1940.

Scaled Scores	English Usage	Study Skills	Reading Comp.	PSYCHOLOGICAL			MATHEMATICS			Scaled Scores	Approximate Percentages
				L	L+Q	Q	I	I+II	II+III		
1	240	90	52	115	180	80	..	60	50	1	3
	208	81	45	92	149	63	..	58	44		
2	207	80	44	91	148	62	30	57	(43	2	7
	197	75	41	84	136	57	..	55	39)		
3	196	74	40	83	135	(56	(29	- (54	38	3	12
	185	69	37	76	127	52)	..)	- 51)	33		
4	184	(68	- (36	- (75	- (126	51	28	50	32	4	18
	171	62)	- 32)	- 69)	- 116)	48	27	46	27		
5	(170	61	31	68	115	47	26	45	26	5	20
	159)	53	27	62	105	43	25	40	22		
6	158	52	26	61	104	42	24	39	21	6	18
	145	45	23	55	94	37	22	35	17		
7	144	44	22	54	93	36	21	34	16	7	12
	134	36	19	49	82	31	19	29	12		
8	133	35	18	48	81	30	18	28	11	8	7
	125	29	16	42	75	21	17	26	9		
9	124	28	15	41	74	20	16	25	8	9	3
	0	0	0	0	0	0	0	0	0		

FIG. 2—Profile chart used at the University of Nebraska to record scores made on pre-registration guidance examinations by entering freshmen, September, 1940. Drawn on the profile chart is the profile of a typical pre-engineering student.

If the profile-chart is drawn for each student by encircling the limiting pairs of raw scores, there is available in graphic form an interpretation of the scores on the guidance examinations. The relatively strong and weak points in the achievement of the student show up as high or low points, respectively, on the profile. Fig. 2 shows the profile of a typical pre-engineering student.

III

Coefficients of correlation were calculated between scores on the classification examination, final examination scores, and semester course marks (see Table 2). The correlations are of a sufficient magnitude to justify the present, initial classification of students. However, the correlational analyses and other studies have suggested instructional problems and problems of curriculum which are being studied by the mathematics staff.

TABLE 2—Coefficients of correlation between scores on 1940 mathematics classification examination and examinations and course marks in freshman mathematics courses. The University of Nebraska.

	Number of Students	Part I	Part II	Part I+II*	Part III	Part II+III*
Examination in Mathematics 11 course	143	.40	.54	.54	....	....
Course Mark in Mathematics 11.	143	.43	.58	.59	....	....
Course Mark in Mathematics 21 —total	160	.47	.55	.59	.45	.56
average†	....	.50	.56	.61	.49	.59

*Multiple correlation coefficients make use of intercorrelations between Parts I and II of the classification examination of .59 and between Parts II and III of .65.

†The weighted average of coefficients of correlations has been obtained in a manner similar to that used in Snedecor's *Statistical Method*, 1937, pp. 220-221.

It would appear from a two-year study that the classification examinations in mathematics identify more clearly and accurately the good students who will succeed than the poor student who will fail early in the sequence of courses. Although a number of factors are operative, student proclivities, on the one hand, and closer agreement among faculty members regarding the qualities of a good student, on the other, deserve consideration in seeking for an explanation of this occurrence. It is obvious that there is closer affinity between the student interested and competent in mathematics and the subject-matter specialist than exists between the less able or less interested student and his teacher. In the studies of the marking practices in the Mathematics Department, for example, the agreement of teachers in the identity of outstanding students has been remarkable, but the less able student whether revealed by the classification examination or by the common final examination has not been so readily categorized by his teachers.

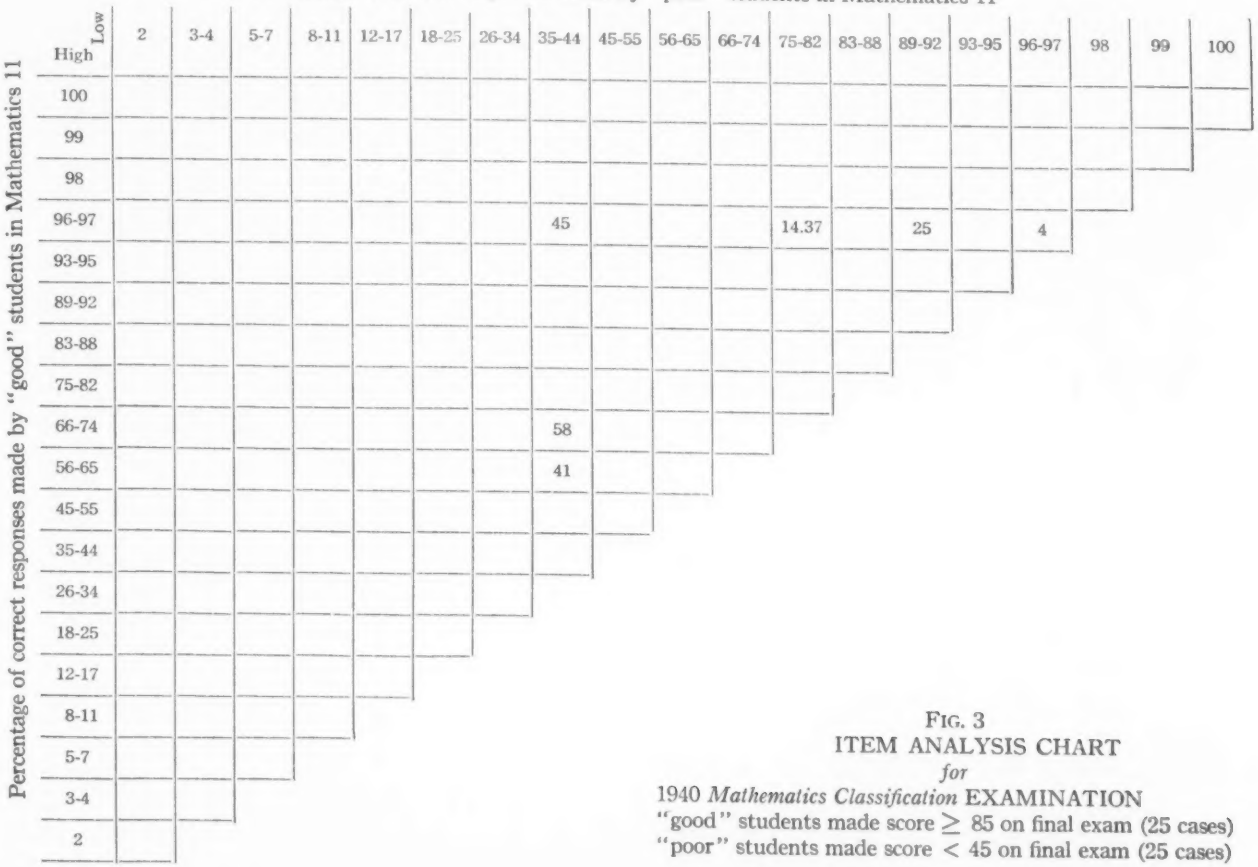

FIG. 3
ITEM ANALYSIS CHART
for
1940 *Mathematics Classification* EXAMINATION
"good" students made score $\geq$ 85 on final exam (25 cases)
"poor" students made score $<$ 45 on final exam (25 cases)

IV

It is not to be assumed that all questions on the examination have contributed to the satisfactory relationships shown in Table 2, nor that they contribute equally well to the criterion of scores of final examinations or of course marks. The percentages of corrrect responses, question by question, have been compared for those students who excelled in mathematics courses at the end of the semester and for those students who made low ratings at the end of the semester. For example, the comparative percentages of correct responses made by "good" and "poor" students in the Mathematics 11 course are shown graphically in Fig. 3 (the criterion used is the final examination score) for questions numbers 4, 14, 25, 37, 41, 45, and 58.* The intervals on the axes, presented as percentages, are adjusted so as to represent intervals of one-fourth of a standard deviation of a *normalized* distribution.

TABLE 3—Percentages of correct responses made by the "good" and the "poor" students in both Mathematics 11 and Mathematics 21 to seven questions of the guidance Mathematics examination.

TEST	MATHEMATICS 11		MATHEMATICS 21	
Item	High	Low	High	Low
Arithmetic				
4	96	96	100	100
14	96	80	94	96
25	96	92	100	80
Algebra				
37	96	76	100	87
41	60	44	97	58
45	96	40	97	80
58	72	36	100	64

It can be readily observed that question 45 differentiates more clearly between the good and poor students than does question 25. Question 4 is an easy question, but it fails to differentiate between the good and poor students. Questions 41 and 58 are more difficult than the other questions appearing on the chart.

It is to be noted in Fig. 3 that only one criterion is applied but that other criteria are mentioned in the discussion. Table 3 gives the percentage of correct responses made by the good and poor students

*The questions listed above are printed in multiple choice form as an appendix to this paper.

in both sequences of freshman mathematics courses on the sampling of questions 14, 25, 37, 41, 45, and 58. It can be observed† that question 14, 37, 45, and 58 discriminate between students in Mathematics 11, that questions 25, 37, 41, and 58 discriminate between students in Mathematics 21, and that questions 4, 41, and 58 discriminate between the two groups of students—Mathematics 11 and Mathematics 21.

V

In the development of a program of student guidance at the University, the mathematics classification examination has been integrated into an all-University measurement program. The extended use of the examination has made necessary changes in the form and scope of the examination. Correlational and statistical analyses have justified the changes on a technical basis, and have, in turn, suggested instructional problems and problems of curriculum which are now being studied by the mathematics staff.

Supplementary references:

Cox, H. M., "A program of instructional research and student guidance," *Journal of the American Association of Collegiate Registrars*, Vol. 17, No. 3, pp. 295-304. (April, 1942).

Cox, H. M. and Harsh, C. M., "Sub-correlation in course marking," *The Journal of Educational Psychology*, Vol. XXXIII, No. 5, pp. 379-384. (May, 1942).

APPENDIX

Sample questions from the 1940 Nebraska Mathematics Examination. (The examination consists of questions of the short-matching type; however, the sample questions are set up in multiple choice form).

4. Divide 6055 by 28. (1) 216.25; (2) 240.91; (3) 241.37; (4) 250.91; (5) 261.25.
14. What is the cube root of 64? (1) 4; (2) 5; (3) 6; (4) 7; (5) 8.
25. If 200 cubic centimeters of water are added to an equal amount of 50% solution of iodine, what will be the per cent of iodine in the mixture? (1) 7%; (2) 25%; (3) 35%; (4) 40%; (5) 70%.
37. Perform operations and simplify: $a(a-b)-b(b-a)$.
(1) $ax+by$; (2) b^2-a^2; (3) $bx-ay$; (4) a^2+b^2; (5) a^2-b^2.
41. What is one of the factors of the expression $4a^3-2a^2b$?
(1) $4a-b$; (2) $4a+b$; (3) $2a-b$; (4) a^3; (5) a^2b.
45. What is the simplified form of the expression $6ax^2y\div(-2ax)$?
(1) $-3xy$; (2) $3xy$; (3) $3x$; (4) 3; (5) -3.
58. What is the meaning of the formula $2\pi r$? (1) Area of a triangle; (2) Circumferference of a circle; (3) Area of a circle; (4) Volume of a cylinder; (5) Pythagorean theorem.

†An extended discussion of the validity of examination questions is beyond the bounds of this paper. A non-technical discussion of this point can be found in *The American Physics Teacher*, Vol. 2, No. 3, Supplement (September, 1934), p. 135.

Notes and Comments

Edited by
N. A. COURT

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4. *Centroid of Moving Points.* Theorem. If n points are moving with arbitrary constant velocities, the centroid will move with constant velocity.

The proof by elementary vector methods is very simple. Let the position of each point be expressed in terms of the parameter t as $p_i = a_i + tv_i$, where a_i is the initial position vector and v_i is the constant vector described in one second. Then the centroid G is clearly given by $\Sigma a_i/n + t(\Sigma v_i/n)$, which completes the proof.

The usefulness of this result can be appreciated by noting that three propositions given by Johnson, *Modern Geometry*, p. 250, can be regarded as corollaries. They are as follows:

If L, M, N divide the sides of the triangle ABC such that $BL/LC = CM/MA = AN/NB$, then the centroid of LMN coincides with that of ABC. For L, M, N can be regarded as starting simultaneously from B, C, A and arriving simultaneously at C, A, B.

If L, M, N, L', M', N' are such that $BL = L'C$, $CM = M'A$, $AN = N'B$, then the line joining the centroids of LMN and $L'M'N'$ is bisected by the centroid of ABC. For points starting simultaneously from L, M, N and arriving simultaneously at L', M', N' will pass through the midpoints of the sides of ABC simultaneously.

If $CM = mBL$, $AN = nBL$, where m and n are arbitrary constants, then the locus of the centroid of LMN is a line through the centroid of ABC. It will be noticed that this statement, a slight generalization of Johnson's article 415, follows directly from the present theorem.

Grinnell College. J. H. BUTCHART.

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5. The evaluation of the indefinite integral

$$\int e^{ax} \sin bx \, dx$$

can be effected readily by the following three steps:

(1) Differentiate the function $e^{ax} \sin bx$,
(2) Change the sign of the new trigonometric function introduced,
(3) Divide the result by a^2+b^2.

This procedure is recommended as a time-saving device, since integrals of this type occur frequently in both pure and applied mathematics.

The same three steps may be followed to evaluate

$$\int e^{ax} \cos bx \, dx.$$

University of Oklahoma. C. E. SPRINGER.

Two Successful Books

FOLEY'S College Physics, 3rd EDITION

This textbook has enjoyed a remarkable reception ever since its publication. Its adaptability to the teaching of physics in the accelerated program is evidenced by many new adoptions. Written in a fluent manner, the text arouses and maintains the student's interest in physics and develops the scientific method of reasoning. By A. L. FOLEY, Indiana University, 470 Illus. 757 Pages. $3.75 (1941).

STRANATHAN'S The "Particles" of Modern Physics

In this book Dr. Stranathan presents a clear and interesting account of the elements of modern physics based upon experimental evidence and practical teaching. By J. D. STRANATHAN, University of Kansas. 211 Illus. 571 Pages. $4.00 (1942).

THE BLAKISTON COMPANY
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Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 461. Proposed by *C. E. Springer*, University of Oklahoma.

Consider a simplex determined by $n+1$ points on a central hyperquadric in n-space, together with the simplex determined by the $n+1$ tangent hyperplanes at these points. Show that the center of the hyperquadric and the centroids of the simplexes either all coincide or are all distinct. (This is an extension of a problem which appears in the *American Mathematical Monthly*, 1936, p. 433.)

Solution by the *Proposer*.

Let the central hyperquadric be represented by $a_{ij}x^ix^j=1$,* with the determinant $|a_{ij}|\neq 0$, $a_{ij}=a_{ji}$ $(i,j=1,\cdots,n)$. Take the points $A_\alpha(\alpha=1,\cdots,n+1)$ with coordinates $y_\alpha{}^i(i=1,\cdots,n)$ on the hyperquadric as vertices of the first simplex. (In what follows, Latin indices will take the range $1,\cdots n$ and Greek indices the range $1,\cdots,n+1$). The tangent hyperplanes at A_α are given by $a_{ij}y_\alpha{}^ix^j=1$ $(\alpha=1,\cdots n+1)$. Take A'_β, with coordinates $z_\beta{}^j$, as the points of intersection of the hyperplanes, A'_β being the point common to the hyperplanes at the points $A_1,\cdots,A_{\beta-1},A_{\beta+1},\cdots,A_{n+1}$.

For any vertex A'_β, we have

$$a_{ij}y_\alpha{}^i\,z_\beta{}^j=1 \qquad (\alpha=1,\cdots,\beta-1,\beta+1,\cdots,n+1). \tag{1}$$

Summing equations (1) relative to α yields

$$\sum_\alpha a_{ij}\,y_\alpha{}^i\,z_\beta{}^j=n \qquad (\alpha\neq\beta). \tag{2}$$

*A repeated index indicates summation.

If the coordinates of the centroid of the simplex with vertices A_α are d^i/n $(i=1,\cdots,n)$, we have

$$\sum_{\sigma=1}^{n+1} y_\sigma{}^i = d^i \qquad \text{or}$$

$$\sum_{\alpha} y_\alpha{}^i = d^i - y_\beta{}^i \qquad (\alpha \neq \beta). \tag{4}$$

Use of (4) in equation (2) gives

$$a_{ij}\, z_\beta{}^j (d^i - y_\beta{}^i) = n \qquad (\beta \text{ not summed}), \tag{5}$$

which may be written in the form

$$a_{ij}\, y_\beta{}^i\, z_\beta{}^j = a_{ij}\, d^i\, z_\beta{}^j - n. \tag{6}$$

The vertices A'_α lie in the hyperplane at A_β for all α except $\alpha=\beta$. Therefore, the $z_\alpha{}^j$ satisfy

$$a_{ij}\, y_\beta{}^i\, z_\beta{}^j = 1 \qquad (\alpha \neq \beta). \tag{7}$$

Summing equations (7) relative to α gives

$$\sum_{\alpha} a_{ij}\, y_\beta{}^i\, z_\alpha{}^j = n \qquad (\alpha \neq \beta). \tag{8}$$

Addition of equations (6) and (8) yields

$$\sum_{\sigma=1}^{n+1} a_{ij}\, y_\beta{}^i\, z_\sigma{}^j = a_{ij} d^i z_\beta{}^j, \tag{9}$$

which can be written in the form

$$a_{ij} y_\beta{}^i \left(\sum_{\sigma=1}^{n+1} z_\sigma{}^i \right) - a_{ij}\, d^i\, z_\beta{}^j = 0. \tag{10}$$

If we require that the centroid of the second simplex coincide with that of the first simplex, equation (10) takes the form

$$a_{ij}\, d^i (y_\beta{}^j - z_\beta{}^j) = 0. \tag{11}$$

Any n of the equations (11) have only the trivial solution $d^i=0$ $(i=1,\cdots,n)$, because the determinant

$$|a_{ij}(y_\beta{}^j - z_\beta{}^j)| = |a_{ij}| \cdot |y_\beta{}^j - z_\beta{}^j| \neq 0.$$

Hence, if the centroids of the simplexes coincide they are at the center of the hyperquadric.

By analogous argument, equation (10) shows that, if $\sum z_\sigma{}^j = 0$, $(j=1,\cdots,n)$, then $d^i=0 (i=1,\cdots,n)$, and conversely.

No. 481. Proposed by *Nelson Robinson*, Norfolk, Virginia.

Find the locus of the center of a circle inscribed to a variable triangle of fixed base and height.

I. Solution by *A. Sisk*, Maryville, Tennessee.

Let A(g,h) be the variable vertex, $B(a,0)$ $C(0,0)$ the fixed vertices of a variable triangle ABC, and let $I(x,y)$ be its incenter. Expressing the tangents of the angles C, B in terms of the tangents of their respective half angles ICB, IBC, we have

$$\frac{h}{g}=\frac{2xy}{x^2-y^2}, \quad \frac{h}{a-g}=\frac{2y(a-x)}{(a-x)^2-y^2}.$$

Eliminating the variable parameter g between the two equations, we obtain

$$2x^2y-h(x^2+y^2)-2axy+ahx=0. \tag{1}$$

This cubic curve passes through the points B and C, and the tangents to the curve at these points are perpendicular to BC. The locus of the incenter is the part of the cubic which lies inside the rectangle formed by the base BC, the tangents to the curve at B, C, and the parallel to BC described by the variable vertex A.

The maximum point on this piece of the cubic has for its coordinates

$$\left(\frac{a}{2}, \quad \frac{a\sqrt{a^2+4h^2}-a^2}{4h}\right),$$

which shows that, in the variable triangle ABC, the incircle is maximum when ABC is isosceles.

II. Solution by *Nev. R. Mind.*

Let B, C be the fixed vertices of a variable triangle ABC, the variable vertex A of which describes a straight line m, not necessarily parallel to the line BC.

As the point A describes the line m, the line BA describes a pencil $B(A,\cdots)$ perspective to the range $(A,\cdots)$, and the two bisectors t_b, t_b' of the angle B describe the involution of orthogonal rays having B for center, and this involution is perspective to the pencil $B(A,\cdots)$. Similarly for the vertex C. Thus we have

$$(t_bt_b',\cdots)\overline{\wedge}B(A,\cdots)\overline{\wedge}(A,\cdots)\overline{\wedge}C(A,\cdots)\overline{\wedge}(t_ct_c',\cdots).$$

Now in the two projective orthogonal involutions of rays the common line BC is a self-corresponding element, hence the locus of

the points of intersection of the pairs $t_b l'_b$, $t_c l'_c$, of corresponding elements of the two involutions lie on a cubic passing through the points B, C.

Further information about the cubic may be obtained by considering special positions of the point A.

Also solved by *Paul D. Thomas* and the *Proposer*. Thomas takes for origin the mid-point M of the fixed base $BC=2a$, and obtains the equation

$$2y(x^2-a^2)=h(x^2+y^2-a^2). \tag{2}$$

He points out that the cubic is also the locus of the excenters of the variable triangle.

EDITORIAL NOTE. Equation (2) shows that the curve is symmetrical with respect to the mediator* MD of the base BC.

The discriminant of (2), considered as a quadratic in y, is

$$4(x^2-a^2)(x^2-b^2), \qquad b^2=a^2+h^2. \tag{3}$$

This discriminant is zero for $x=\pm a$, $x=\pm b$, and the corresponding values of y are 0 and h, whence the tangents to the curve at the points $B(-a,0)$ $C(a,0)$ $E(b,h)$, $F(-b,h)$ are parallel to the Y-axis.

The discriminant (3) is negative when the absolute value of x is between a and b, whence no points of the cubic lie between the parallels to the Y-axis through the points B and F.

The asymptotic directions of the curve are obtained by equating to zero the terms of the highest degree, whence these directions are $x=0$, $y=0$.

The y-coordinates of the points in which the cubic is met by the line $x=c$, where c is an arbitrary constant, are given by the equation

$$2y(c^2-a^2)=h(c^2+y^2-a^2).$$

The coefficient of y^2 does not involve c, whence there is no value of c for which that coefficient would be zero, i. e., the curve has no asymptote parallel to the Y-axis, and in the direction of the Y-axis the curve is tangent to the line at infinity.

Again, the x-coordinates of the points of intersection of the cubic with the line $y=c$, where c is an arbitrary constant, are given by the equation

$$(2c-h)x^2=2a^2c+c^2h-a^2h.$$

*i. e. The perpendicular bisector of BC.

For $c=\frac{1}{2}h$ both roots of this equation become infinite, whence the line $y=\frac{1}{2}h$ is an asymptote of the cubic, and, moreover, its point of contact is a point of inflection of the curve.

From these considerations it is apparent that the cubic considered consists of three separate branches. One branch is a closed oval symmetrical with respect to the Y-axis, passing through B, C, with tangents at these points parallel to the Y-axis. The other two branches lie above the line $y=\frac{1}{2}h$, which line they have for an asymptote. The two branches are symmetrical with respect to the Y-axis: they pass through the points E, F, respectively, and the tangents at those points are parallel to the Y-axis.

Geometric considerations afford a rapid way of constructing the cubic by points.

Let a circle (O) passing through B, C meet the given parallel EF to BC in the points A, A', and the mediator MD of BC in the points K, K', the point K' and the line EF lying on the same side of BC. The lines AK, AK' are the internal and external bisectors of the angle A of the triangle ABC, and similarly for the line $A'K$, $A'K'$, with regard to $A'BC$.

The circle (K) having K for center and $KB=KC$ for radius cuts AK in the incenter I and the excenter I_a relative to the side BC of the triangle ABC, and cuts $A'K$ in the analogous points I', I_a', for the triangle $A'BC$. Similarly the circle (K') having K' for center and $K'B=K'C$ for radius cuts $K'A$ in the excenters I_b, I_c relative to the sides AC, AB of ABC, and cuts $A'K'$ in the excenters I_b', I_c' of $A'BC$. The points lying on each of the two circles (K), (K') form a rectangle, the lines II', I_aI_b', I_bI_c', I_cI_b' being parallel to the base BC, and the lines II_a', $I'I_a$, $I_bI'_b$, I_cI_c' being perpendicular to BC. By making the circle (O) vary so as to describe the coaxal pencil having B, C for basic points, the cubic may be constructed by points.

The smallest circle of this pencil which yields points on the cubic is the one which is tangent to the given line EF at the point D where EF meets the mediator MD of the segment BC. It is readily seen that in this case the points I_b, I_b' coincide with the point F, the points I_c, I_c' coincide in E, the points I, I' coincide in a point P on the mediator MD, and so do the points I_a, I_a' coincide in a point Q on MD, the points P and Q being the highest and lowest points on the oval constituting the locus of the points I, I_a.

The remainder of the discussion offers no difficulty. Moreover it becomes clear that one of the infinite branches of the cubic is the locus of the excenter I_b relative to the variable side CA of ABC, and the other infinite branch is the locus of the excenter I_c. The part of the

oval lying between the base BC and the line EF is the locus of the incenter I, while the remainder of the oval is the locus of the excenter I_a relative to the fixed base BC.—N. A. C.

No. 484. Proposed by *H. C. D. McCluskey*, student, Colgate University.

A cylinder, without top, weighs m grams. Its center of gravity is assumed to be c centimeters below its center. To what depth must water be added to bring the center of gravity of the combined mass to its lowest possible point?

Solution by the *Proposer*.

Let the height of the can be h. Then its center of gravity is at the distance $\frac{1}{2}h-c$ centimeters above the bottom. Let y be the depth of the water which is added, and let Ky be the weight of the water in grams. Further, let $\bar{y}$ be the distance above the bottom of the centroid of the combined mass. By the principles of moments we have the relation

$$\bar{y}=\frac{m(\frac{1}{2}h-c)+Ky^2/2}{Ky+m}, \tag{1}$$

whence

$$\frac{d\bar{y}}{dy}=\frac{(Ky+m)Ky-[m(\frac{1}{2}h-c)+Ky^2/2]K}{(Ky+m)^2}. \tag{2}$$

$\bar{y}$ will be a minimum when the numerator of (2) vanishes. That is when

$$(Ky+m)y=m(\tfrac{1}{2}h-c)+Ky^2/2, \tag{3}$$

whence the explicit expression for y is easily obtained. Comparison of (1) with (3) shows, as might have been expected, that the centroid of the combined mass reaches its lowest point at the instant when it lies in the surface of the water.

No. 486. Proposed by *V. Thébault*, Tennie, Sarthé, France.

In what system of numeration having a base less than 10000 are there the largest number of 4-digit squares of the form *aabb* and such that $aabb=(cc)^2$?

EDITORIAL NOTE. The *Proposer* stated that for the base 9241, there are 63 squares of the proposed form: $N_1=\overline{19}\ \overline{19}\ \overline{9223}\ \overline{9223}=(\overline{419}\ \overline{419})^2$, $N_2=\overline{23}\ \overline{23}\ \overline{9219}\ \overline{9219}=(\overline{761}\ \overline{761})^2$, $\cdots$.

The result may be obtained by the following analysis. If B is the base, the hypothesis leads to the relation

$$(aB^2+b)(B+1)=c^2(B+1)^2$$

whence we obtain

$$c^2=\frac{aB^2+b}{B+1}=aB-a+\frac{a+b}{B+1}.$$

Since $0<a+b<2B$, we must have $(a+b)/(B+1)=1$, and thus $a=(c^2-1)/(B-1)$. Hence $c<B-1$, and c must satisfy the congruence

$$c^2\equiv 1 \mod B-1. \tag{1}$$

Let $B-1=2^r p_1^{r_1}\cdots p_m^{r_m}$ where p_i are distinct odd primes. Then (1) is satisfied by all simultaneous solutions of the system

$$c\equiv\pm 1 \mod p_1^{r_1},\cdots, \quad c\equiv\pm 1 \mod p_m^{r_m}, \quad c\equiv\pm 1 \mod 2^{r-1},$$

(The modulus of the last congruence is changed to 2 when $r=1$, and the congruence is omitted when $B-1$ is odd.) From elementary number theory we know that this system has 2^{m+2} solutions for $c<B-1$, when $r>2$; 2^{m+1} solutions when $r=2$; 2^m solutions when $r=0,1$. From these totals, the solution $c=1$ must be excluded, since $a\neq 0$. For $B=9241$, $B-1=2^3\cdot 3\cdot 5\cdot 7\cdot 11$ gives $64-1$ squares of the desired form, evidently the largest number possible without introducing another odd prime factor and raising the value of B above 10000.—E. P. S.

PROPOSALS

No. 523. Proposed by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

Find the power of the incenter of a triangle with respect to the circumcenter, in terms of the sides of the triangle.

No. 524. Proposed by *J. Frank Arena*, Hardin, Illinois.

The equation $ax+by=c$ is to have exactly n distinct pairs of positive integral solutions (x,y). For given a, b, show that the greatest value of c is $(n+1)ab-a-b$, and that the least value of c is

$$(n-1)ab+a+b.$$

No. 525. Proposed by *V. Thébault*, Tennie, Sarthé, France.

Two lines drawn through the vertex A of a parallelogram $ABCD$ and symmetric with respect to the bisector of the angle A, meet the

sides BC and CD in the points E, F and G, H. Show that the points E, F, G, H, lie on a circle, and determine the locus of the center of that circle when the angle between the two given lines varies.

No. 526. Proposed by *S. W. Reaves*, University of Oklahoma.

An innkeeper wishes to seat seven guests at a round table for a series of dinners in as many ways as possible, subject to the condition that at no two dinners shall any guest have the same two adjacent neighbors. Determine the seating arrangements.

No. 527. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Construct a triangle ABC, given a, $b+c$, and $(p-q)/h_a$, where p and q are the projections of the sides b and c upon the side a.

No. 528. Proposed by *N. A. Court*, University of Oklahoma.

The two sums of the squares of the distances of the vertices of two twin tetrahedrons from any given point in space are equal.

No. 529. Proposed by *William Taylor*, student, Colgate University.

Consider a field piece firing shells at a constant muzzle velocity but with varying angle of elevation. Prove that the vertices of the paths of these shells form an ellipse. (Neglect air resistance.)

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Analytic Geometry. By Edward S. Smith, Meyer Salkover, and Howard K. Justice. John Wiley & Sons, New York, 1943. xii+298 pages.

This text is written by three professors from the University of Cincinnati. It is written with satisfying clarity of statement and rigor throughout, yet it is not too difficult for the first-year college level.

The first chapter covers very clearly the usual tool definitions and formulas of plane analytics. The authors do well in giving to the student the ideas of sense and directed segments. It was a good choice to include a thorough explanation of the range of a variable in this chapter.

The topics covered in Chapter II are the fundamental problems: the "Locus, Equations, and the Graph".

Chapter III is devoted to the straight line. This chapter shows no unusual approach.

Chapter IV is not a usual chapter. It treats the circle, ellipse, parabola, and hyperbola as sections of a right circular cone. By showing these curves as sections of a cone and stating their basic definitions in terms of the cone, the authors introduce the student to the next four chapters which cover the *Circle*, *Ellipse*, *Parabola*, and *Hyperbola*, respectively. The figures of Chapter IV are excellent.

The four chapters on the circle, ellipse, parabola, and hyperbola include the usual material, largely. The interesting construction of the hyperbola from its asymptotes and a known point on the curve in Section 62 is not ordinarily included in an elementary text on analytics.

Transformation of Coordinates, *The Equation of the Second Degree*, *Tangents and Normals*, *Higher Plane Curves*, *Polar Coordinates*, *Parametric Equations*, and *Empirical Equations*, are respectively, the topics of the last seven chapters of the plane analytics in the text. The use of the parameter, k, in Section 76 for the determination of the general conic through five points and the parabola through four points will be of special interest to the brighter student.

The text closes with five chapters on solid analytics. An appendix on the proof of the invariance of the determinant Δ in the theory of the general equation of the second degree is given. Also included in the back of the book is a set of formulas for reference, a table of trigonometric functions of special angles, a four-place table of common logarithms, a table of natural logarithms, a table of values of e^x and e^{-x}, and answers to the odd problems.

This text is carefully written, the figures are splendid, and the format is pleasing. The text is recommended for examination to any mathematics department planning a change in analytic geometry texts.

Louisiana Polytechnic Institute. P. K. SMITH.

An Introduction to Managerial Business Statistics. By Harry Pelle Hartkemeier. Thomas Y. Crowell Company, New York, 1943. xiv+207 pages. $1.75.

The author states in the preface that this book is a first course in statistics and that it can be understood by students who have not had any college mathematics. After a short introduction the author, in the second chapter of 24 pages, discusses briefly the following: arithmetic average, standard deviation, normal curve, comparison of two frequency distributions, degrees of freedom, standard error of a mean, comparison of two averages, standard error of the difference between two averages, the standard error of the standard deviation, the F-ratio, probability, the t-ratio and a table of values of F, t, and X^2. These two dozen pages contain statements about the most fundamental ideas in statistics. Usually at least two full terms are required to present the necessary material for students to understand the material in this chapter. Students and teachers using this book will have to secure other statistical texts to understand the material covered; for they cannot possibly comprehend these fundamental statistical concepts from what is given.

Chapter 3 presents a good discussion pertaining to the null hypothesis. At the end of this division are some questions which have no relation to the subject matter presented. Here it is assumed that the student is familiar with correlation, fiducial limits, index numbers, sampling theory, time series and information which can be obtained by a thorough course in statistics.

The next chapter deals with computing procedure and machines without introducing the reader to any of the modern calculating machines. Most of this section is devoted to rounding off figures before and after squaring.

Chapter five introduces ideas concerning chi-squares for 2-way tables together with charts depicting areas "under" X^2-curves.

The last chapter contains material pertaining to production control, which is illustrated with control charts for averages and standard deviations and a normal curve and table.

At the end of each chapter are problems.

The appendix contains tables of squares, square roots, and reciprocals of numbers.

Of the 207 pages in the book, 55 are blank pages and 21 are form pages set up for problems or answers to questions. These form pages contain formulas into which the student can substitute numerical values. Of the remaining 131 pages, all but 78 are devoted to charts, tables, and the index.

From the last sentence, it appears as though this book is to be followed by three other statistical books. It is hoped that they clarify most of the material mentioned in the first volume.

Michigan State College. W. D. BATEN.

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